20/Aug/2009

Manjunath
Krishnapur

Theorem: [Paley-Wiener-Zygmund] $(B_t)_{0\le t\le 1}$ - std. 1 dimensional BM. Fix $\alpha<\frac{1}{2}$.

Then
$$P\left\{ \sup_{s\neq t} \frac{|B_t - B_s|}{|t-s|^\alpha} < \infty \right\} = 1$$

Proof: Step 1: For any $n\geq 1$ and $1\le k\le 2^n$, $\left(B\left(\frac{k+1}{2^n}\right) - B\left(\frac{k}{2^n}\right)\right) \sim N(0, 1/2^n)$.

Hence
$$P\left\{ \left|B\left(\tfrac{k+1}{2^n}\right) - B\left(\tfrac{k}{2^n}\right)\right| > \frac{1}{2^{n\alpha}} \right\} \le P\{ |X| > 2^{n(1/2-\alpha)} \} \quad (X\sim N(0,1))$$
$$\le e^{-2^{n(1-2\alpha)}\cdot\frac{1}{2}}$$

$$\Rightarrow \quad P\left\{ \max_{1\le k\le 2^n} \left|B\left(\tfrac{k+1}{2^n}\right) - B\left(\tfrac{k}{2^n}\right)\right| > \frac{1}{2^{n\alpha}} \right\} \le 2^n e^{-2^{n(1-2\alpha)-1}}$$

which is summable, ie,
$$\sum_n P\left\{ \max_{1\le k\le 2^n} \left|B\left(\tfrac{k+1}{2^n}\right) - B\left(\tfrac{k}{2^n}\right)\right| > \frac{1}{2^{\alpha n}} \right\} < \infty$$

$\Rightarrow$ $\exists\, N(\omega)$ s.t. $P\{N(\omega)<\infty\} = 1$ and
$$\left|B\left(\tfrac{k+1}{2^n}\right) - B\left(\tfrac{k}{2^n}\right)\right| \le \frac{1}{2^{\alpha n}} \qquad \forall\, n\geq N(\omega),\ \forall\, 1\le k\le 2^n$$

Set
$$C(\omega) = \max_{n\le N(\omega)} \cdot \max_{1\le k\le 2^n} \cdot 2^{\alpha n} \left|B\left(\tfrac{k+1}{2^n}\right) - B\left(\tfrac{k}{2^n}\right)\right|$$

so that

$$(\star) \quad - \quad P\{C(\omega)<\infty\} = 1 \quad \text{and} \quad \left|B\left(\tfrac{k+1}{2^n}\right) - B\left(\tfrac{k}{2^n}\right)\right| \le \frac{C(\omega)}{2^{\alpha n}} \quad \forall n\geq 1,\ \forall 1\le k\le 2^n$$

Step 2: Now consider any $0\le t<s\le 1$.

Let $2^{-m_0} < s-t < 2^{-m_0+1}$. Then there is atleast one (and atmost two) dyadics $u_0 = \frac{k_0}{2^{m_0}} \in [s,t]$

Fix one such u_0

Consider the interval $[u_0, s]$.

$$u_0 = \frac{k_0}{2^{m_0}} = \frac{2^\ell k_0}{2^{m_0+\ell}} \quad \text{for } \ell = 0,1,2,\dots$$

Then pick the smallest $\ell \geq 0$ such that ~~the~~

$$\frac{2^\ell k_0 + 1}{2^{m_0+\ell}} \leq s$$

Define $u_1 = \frac{k_1}{2^{m_1}}$ where $m_1 = m_0 + \ell$, $k_1 = 2^\ell k_0 + 1$

Note that $m_1 \geq m_0$ $(\ell \geq 0)$ and equality could hold at this first step.

Next note that $u_1 = \frac{2^\ell k_1}{2^{m_1+\ell}}$ $\forall\, \ell = 0,1,2,\dots$

and pick the smallest $\ell \geq 0$ such that $\frac{2^\ell k_1 + 1}{2^{m_1+\ell}} \leq s$

Set $u_2 = \frac{k_2}{2^{m_2}}$ where $m_2 = m_1 + \ell$, $k_2 = 2^\ell k_1 + 1$

Now we claim that $\ell \geq 1$ and hence $m_2 > m_1$. This is because, if $\ell = 0$ then $u_1 = \frac{k_1}{2^{m_1}}$, $u_2 = \frac{k_2}{2^{m_1}}$ and by picking an even number $k_1 \leq 2j \leq k_2$ we get a dyadic $\frac{2j}{2^{m_1}} = \frac{j}{2^{m_1-1}}$ which is between u_0 and s. This means we ~~could~~ could have picked a dyadic with smaller m_1 in the first step (Needs some thought)

Thus, successively, having picked $u_j = \frac{k_j}{2^{m_j}}$, $1 \leq j \leq p-1$, we ~~pick~~ write

$$u_{p-1} = \frac{2^\ell k_{p-1}}{2^{m_{p-1}+\ell}} \quad \text{for } \ell = 0,1,2,\dots$$ and pick the smallest ℓ

for which $\frac{2^\ell k_{p-1}+1}{2^{m_{p-1}+\ell}} \leq s$. We then set

$$u_p = \frac{k_p}{2^{m_p}} \quad \text{where} \quad k_p = 2^\ell k_{p-1} + 1, \quad m_p = m_{p-1} + \ell$$

If $\ell = 0$, then it means that one of u_{p-1} or u_p ~~for something m~~

is a dyadic of denominator $m_{p-1}-1$ which contradicts our choice of u_{p-1} ..... (needs some thought!)

Thus we get $t \le u_0 < u_1 < \dots \le s$ (the procedure terminates if $u_p = s$ for some p). Either the procedure terminates or $u_p \to s$ as $p \to \infty$.

Step 3: ~~$B_s - B_{u_0}$~~ $B_\omega(s) - B_\omega(u_0) = \sum_{k=1}^{\infty} B(u_k) - B(u_{k-1})$

and for each j, $u_{j-1} = \frac{k_{j-1}}{2^{m_{j-1}}} = \frac{l_{j-1}}{2^{m_j}}$ $\left(l_{j-1} = k_{j-1} 2^{m_j - m_{j-1}}\right)$

and $u_j = \frac{k_j}{2^{m_j}}$ where $k_j = l_{j-1} + 1$

Therefore, by $\circledast$), we get $\left|B_\omega(u_j) - B_\omega(u_{j-1})\right| \le \frac{c(\omega)}{2^{\alpha m_j}}$

Hence, $\left|B_\omega(s) - B_\omega(u_0)\right| \le \sum_{j=1}^{\infty} \frac{c(\omega)}{2^{\alpha m_j}}$

$$\le \frac{c(\omega)}{2^{m_0 \alpha}} \left(1 + \frac{1}{2^\alpha} + \frac{1}{2^{2\alpha}} + \dots\right)$$

because $m_1 < m_2 < m_3 < \dots$

$$= \frac{c'(\omega)}{2^{m_0 \alpha}} \qquad \left(\because c'(\omega) = \frac{c(\omega)}{1 - 1/2^\alpha}\right)$$

$$\le c'(\omega)(s-t)^\alpha \qquad \left(\because 2^{-m_0} < s-t\right)$$

Analogously, one can prove that $\left|B_\omega(u_0) - B_\omega(t)\right| \le c'(\omega)(s-t)^\alpha$

Thus, $\left|B_\omega(s) - B_\omega(t)\right| \le 2c'(\omega) \cdot (s-t)^\alpha$

This is true for all s, t. Hence the theorem is proved. ∎

Remark: Suppose we try to prove that $P\left\{ \sup_{\substack{s \ne t \\ s < t}} \frac{|B_t - B_s|}{\sqrt{(t-s)\log\left(\frac{1}{t-s}\right)}} < \infty \right\} = 1.$

P.T.O.

Then, the first step changes a little and we get

$$P\left\{ \left| B\left(\tfrac{k+1}{2^n}\right) - B\left(\tfrac{k}{2^n}\right) \right| > \frac{c\sqrt{\log 2^n}}{\sqrt{2^n}\ \cancel{\sqrt{\log 2^n}}} \right\}$$

$$= P\left\{ |X| > c\sqrt{\log 2^n} \right\} \leq e^{-\frac{c^2}{2}\log 2^n}$$

~~Hence~~ $\cancel{2^n P\{}$ $2^n e^{-\frac{c^2}{2}\log 2^n}$ is summable if $c > \sqrt{2}$

and we get the following analogue of $(*)$

For $c > \sqrt{2}$, $\exists N(\omega)$ s.t. $P\{N(\omega) < \infty\} = 1$ and

$$(**) \quad \left| B\left(\tfrac{k+1}{2^n}\right) - B\left(\tfrac{k}{2^n}\right) \right| \leq \frac{c\sqrt{\log 2^n}}{\sqrt{2^n}} \quad \forall\, n \geq N(\omega)$$

One can then work the same way as before to prove

that $\quad |B(t) - B(s)| \leq c' \sqrt{(t-s)\log\left(\tfrac{1}{t-s}\right)} \quad \forall\, 0 \leq s < t \leq 1$

for some (random) constant c'

This is one half of (a weaker version of) Levy's modulus of continuity.